Math 261B Thurs. Nov. 5

$PSL_2 \cong SO_3$

$O_2(SO_3)$

Chevalley basis $\{x^2, 2xy, y^2\}$ in $V^2 \curvearrowleft PSL_2$

Discriminant $ac-b^2$, or $Q(v) = x_1x_3 - x_2^2$

$= $ invariant form

$Q(Av) = Q(v)$

$+ \ \det(A) = 1$

$$a_{11}a_{31} - a_{21}^2 = 0 \qquad a_{11}a_{32} + a_{31}a_{32} - 2a_{21}a_{22} = 0$$

$$a_{12}a_{32} - a_{22}^2 = -1 \qquad a_{11}a_{33} + a_{31}a_{33} - 2a_{21}a_{23} = 1$$

$$a_{13}a_{33} - a_{23}^2 = 0 \qquad a_{12}a_{33} + a_{32}a_{33} - 2a_{22}a_{23} = 0$$

$$\Downarrow$$

$so_3/\mathbb{Z}$ $\quad A = I + \varepsilon M$

$+ \ \operatorname{tr} M = 0$

$$M_{31} = 0 \qquad M_{32} - 2M_{21} = 0$$

$$2M_{22} = 0 \qquad M_{11} + M_{33} = 0$$

$$M_{13} = 0 \qquad M_{12} - 2M_{23} = 0$$

$$M = \begin{pmatrix} x & 2y & 0 \\ z & 0 & y \\ 0 & 2z & -x \end{pmatrix}$$

$sl_2 \curvearrowright V^2$ on Chevalley basis

$$\begin{tikzcd}
x^2 \arrow[r, bend right, "1"'] \arrow[loop below, "2"] & 2xy \arrow[l, bend right, "2"'] \arrow[r, bend right, "2"'] \arrow[loop below, "0"] & y^2 \arrow[l, bend right, "1"'] \arrow[loop below, "-2"]
\end{tikzcd}$$

$E = x\,\partial/\partial y$

$F = y\,\partial/\partial x$

$H = x\,\partial/\partial x - y\,\partial/\partial y$

$$E = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \qquad H/2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$so_3 = \mathbb{Z} \cdot \{E, F, H/2\}$$

$(PSL_2$ vs. $SL_2)$

$[H/2, E] = E \qquad [H/2, F] = -F$

$[E, F] = 2(H/2)$

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{Z}}(T) & T \subset SL_2 & \\ \uparrow & \downarrow & \\ \mathcal{O}_{\mathbb{Z}}(T') & T' \subset PSL_2 & \end{array}$$

$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ $\quad \mathcal{O}_{\mathbb{Z}}(T) = \mathbb{Z}[t^{\pm 1}]$

$T' = T/\{\pm 1\} \quad \mathcal{O}_{\mathbb{Z}}(T') = \mathbb{Z}[t^{\pm 2}]$

$H \in \mathcal{U}_{\mathbb{Z}}(t) \qquad H/2 \in \mathcal{U}_{\mathbb{Z}}(t')$

$$\begin{array}{ccc} SL_2 & \longrightarrow & SO_3 \\ & \searrow \quad \nearrow & \\ & PSL_2 & \\ & \| & \\ & SL_2/\{\pm 1\} & \end{array}$$

$$\begin{array}{c} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \cap \\ SL_2 \end{array} \longmapsto \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix} = B$$

$ad - bc = 1$

$x \mapsto ax + cy$

$y \mapsto bx + dy$

$x^2 \mapsto a^2x^2 + ac\, 2xy + c^2y^2$

$2xy \mapsto 2ab\, x^2 + (ad+bc)2xy + 2cd\, y^2$

$y^2 \mapsto b^2x^2 + bd\, 2xy + d^2y^2$

$PSL_n = PGL_n$

$$\mathcal{O}_{\mathbb{Z}}(PGL_n) \quad\quad \mathbb{Z}[a_{11}, \dots, a_{nn}, \det(A)^{-1}]$$

$$\|$$

$$\mathbb{Z}[\underline{a}, \det(A)^{-1}]_0 \quad \swarrow \mathbb{Z} \text{ grading} \quad\quad \uparrow \; \mathcal{O}_{\mathbb{Z}}(GL_n)$$

$\mathbb{Z}[\underline{x}, f^{-1}]_0$

$\cong$

$(\mathbb{Z}[\underline{x}]/(f-1))_{d\mathbb{Z}}$

$\deg(f) = d$

$$\cong \quad \downarrow$$

$$(\mathbb{Z}[\underline{a}]/(\det(A)-1))_{n\mathbb{Z}} \quad\quad \frac{f(\underline{a})}{\det(A)^k} \leftarrow \text{homogeneous of degree } kn$$

$$\|$$

$$\mathcal{O}_{\mathbb{Z}}(PSL_n) \quad\quad \nwarrow \mathbb{Z}/n\mathbb{Z} \text{ grading}$$

$$PSL_2 \to SO_3 \quad \rightsquigarrow \quad \mathcal{O}_{\mathbb{Z}}(PSL_2) \longleftarrow \mathcal{O}_{\mathbb{Z}}(SO_3)$$

$$\|$$

$$\left(\mathbb{Z}[a,b,c,d]\Big/\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}-1\right)_{2\mathbb{Z}}$$

$a_{11}, \dots, a_{33}$, eqn's

$Q(Av) = Q(v)$

$\det(A) = 1$

ij entry of B $\longleftarrow$ a_{ij}

(prev. slide)

$PSL_2 \xrightarrow[?]{\simeq} SO_3$

10 Quadratic monomials a^2, ab, ... d^2 generate $\mathcal{O}_{2}(PSL_2)$

$ad - bc = 1$

(9 independent)

Get a^2, ac, c^2, b^2, bd, b^2, $\overset{?}{2ab}$, $\overset{?}{2cd}$

$ad + bc$

$2ad = (ad+bc) + (ad-bc)$

$= ad + bc + 1$

$2bc = ad + bc - 1$

Where are ab, cd, ad, bc ??

$ab = ab(ad-bc) = a^2bd - ab^2c = a^2(bd) - b^2(ac)$ ✓

$cd = cd(ad-bc) = acd^2 - bd\,c^2$ ✓

$ad = ad(ad-bc) = a^2d^2 - abcd$ ✓

$bc = (ad+bc) - ad.$

$$\mathcal{O}_{\mathbb{Z}}(SO_3) \twoheadrightarrow \mathcal{O}_{\mathbb{Z}}(PSL_2) \qquad \cong$$

$\hookrightarrow$ (trivial)

$\mathrm{Spec}(R) \to G/\mathbb{Z}$

$R \leftarrow \mathcal{O}_{\mathbb{Z}}(G)$

Given $\mathcal{O}_{\mathbb{Z}}(G)$, get $R \to \underline{G}(R)$

$\mathcal{O}_{\mathbb{Z}}(G) = \mathbb{Z}[\underline{a}]/I$

$= \{$ ring homomorphism $\mathcal{O}_{\mathbb{Z}}(G) \to R\}$

$= \{$ solutions in R of eqn's of $G\}$

over $\mathbb{Z}$

functor (comm. rings) $\to$ groups

$\mathcal{O}_{\mathbb{Z}}(G)$ is a Hopf algebra

$g, h \in \underline{\underline{G}}(R)$ $\qquad$ $g, h : \mathcal{O}_{\mathbb{Z}}(G) \to R$

$R = \mathbb{F}_q$ finite field.

$\underline{\underline{G}}(\mathbb{F}_q)$ is a <u>finite Chevalley group</u>

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{Z}}(G) \arrow[r, "\Delta"] \arrow[rdd, "gh"'] & \mathcal{O}_{\mathbb{Z}}(G) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{Z}}(G) \arrow[d, "g \otimes h"] \\
& R \otimes_{\mathbb{Z}} R \arrow[d, "\text{multiply}"] \\
& R
\end{tikzcd}$$

$G = GL_n \longrightarrow GL_n(\mathbb{F}_q)$

$SL_n \longrightarrow SL_n(\mathbb{F}_q)$

$SO_N \longrightarrow SO_N(\mathbb{F}_q) \subset SL_n(\mathbb{F}_q)$ preserves Q

$Sp_{2n} \longrightarrow Sp_{2n}(\mathbb{F}_q)$

$PGL_n \longrightarrow PGL_n(\mathbb{F}_q) = GL_n(\mathbb{F}_q) / \mathbb{F}_q^\times \cdot I$ $\qquad$ (Lucky!)

$\shortparallel$

PSL_n $\qquad$ $SL_n(\mathbb{F}_q) / \mu_n(\mathbb{F}_q) \hookrightarrow GL_n(\mathbb{F}_q) / \mathbb{F}_q^\times$

not $\twoheadrightarrow$

$\det(c I_n) = c^n$ $\qquad$ $\mathcal{O}_{\mathbb{Z}}(\mu_n) = \mathbb{Z}[x] / (x^n - 1)$

$SL_n(\mathbb{F}_q)$ $\quad GL_n(\mathbb{F}_q) \xrightarrow{\det} \mathbb{F}_q^\times$

$SL_n(\mathbb{F}_q) / \mu_n(\mathbb{F}_q) \to PGL_n(\mathbb{F}_q) \xrightarrow[\text{"det"}]{} \underbrace{\mathbb{F}_q^\times / (n^{th} \text{ powers in } \mathbb{F}_q^\times)}_{\neq 1 \text{ in general}}$

$\quad\quad = GL_n(\mathbb{F}_q)/\mathbb{F}_q^\times$

$SL_n/\mu_n \rightsquigarrow SL_n(\mathbb{F}_q)/\mu_n$

$\wr$

simply $\underline{\text{connected}}$, semisimple, indecomposable, mod center G

$\underline{\underline{G}}(\mathbb{F}_q) / \underline{\underline{Z(G)}}(\mathbb{F}_q) \longleftarrow$ is usually a finite simple group

$\mathbb{F}_q \subset K = \overline{K}$

$\underline{\underline{G}}(K) \quad\quad \mathcal{O}_K(G) = K \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{Z}}(G)$

$\to$ relative frobenius morphism

$F \quad a_{ij} \mapsto a_{ij}^q$

$G \quad K[a_{ij}]/I$

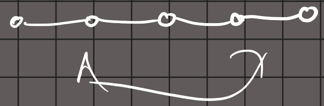
F | over k $\downarrow$ $\mathbb{Z}$ coeff's $\dashrightarrow$ $\mathbb{F}_p$

G $\quad K[a_{ij}^{q}]/I$

$\mathbb{F}_q = K^F$

$G \xrightarrow[F]{} G$ $\qquad \underline{\underline{G}}(\mathbb{F}_q) = (G_K)^F$

more generally take "F" to be (rel Frobenius) $\circ$ (diagram automorphism)



$\leftrightarrow$ GL_n $\circlearrowleft$

$g \mapsto (g^R)^{-1}$

$\mathbb{F}_{q^2} \xrightarrow{\;\overline{(\cdot)}\;} \mathbb{F}_{q^2}$
$\cup$
$\mathbb{F}_q$

$\mathbb{C}$
$\cup$
$\mathbb{R}$

$GL_n(k)^F \subset GL_n(\mathbb{F}_{q^2})$

$\uparrow$

analog of U_n

$(g^R)^{-1} = \overline{g}$

(or SU_n)